1.5 Integration durch Partialbruchzerlegung

Beispiel:

$$\int \frac{2x^3 - 14x^2 + 14x + 30}{x^2 - 4}\,dx = ?$$

1. Schritt: Polynomdivision (nur dann, wenn Grad des Zählers > Grad des Nenners)

$$\left(2x^3 - 14x^2 + 14x + 30\right) : \left(x^2 - 4\right) = 2x - 14 + \frac{22x - 26}{x^2 - 4}$$

$$\begin{array}{l}
\underline{-2x^3 \qquad\quad -8x} \\
\qquad -14x^2 + 22x + 30 \\
\underline{- \quad -14x^2 \qquad\quad +56} \\
\qquad\qquad\quad 22x - 26
\end{array}$$

Die Division ist beendet, weil der Grad des Ergebnisses kleiner als der des Nenner-Polynoms ist.

2. Schritt: Bestimmung der Nullstellen des Nenner-Polynoms

$$x^2 - 4 = 0 \Rightarrow x_{1,2} = \pm 2$$
$$x^2 - 4 = (x-2)\cdot(x-(-2))$$

3. Schritt Aufstellen des Ansatzes für die Partialbruchzerlegung

$$\frac{22x+26}{x^2-4} = \frac{22x-26}{(x-2)\cdot(x+2)} = \frac{A}{(x-2)} + \frac{B}{(x+2)} \left|\, \cdot x^2 - 4 = (x-2)\cdot(x+2)\right.$$

4. Schritt: Bestimmung der Unbekannten Konstanten

$$22x - 26 = A\cdot(x+2) + B\cdot(x-2)$$

$$x = 2: \quad 22 - 2 - 26 = A\cdot(2+2) + B\cdot(2-2)$$
$$\Rightarrow 18 = 4A \Leftrightarrow A = \frac{9}{2}$$
$$x = -2: 22 - (-2) - 26 = A\cdot(2-2) + B\cdot(2-(-2))$$
$$\Rightarrow -70 = -4B \Leftrightarrow B = \frac{35}{2}$$

$$\frac{2x^3 - 14x^2 + 14x + 30}{x^2 - 4} = 2x - 14 + \frac{\frac{9}{2}}{(x-2)} + \frac{\frac{35}{2}}{(x+2)}$$

5. Schritt Integration

$$\int 2x-14+\frac{\frac{9}{2}}{x-2}+\frac{\frac{35}{2}}{x+2}\,dx$$
$$=2\cdot\int x\,dx-14\cdot\int 1\,dx+\frac{9}{2}\cdot\int\frac{1}{x-2}\,dx+\frac{35}{2}\cdot\int\frac{1}{x+2}\,dx$$
$$=x^2-14x+\frac{9}{2}\cdot\ln\left(\left|x-2\right|\right)+\frac{35}{2}\cdot\ln\left(\left|x+2\right|\right)+c$$

Anmerkung:

$$\int\frac{1}{x}\,dx=\ln\left(\left|x\right|\right)+c$$

$$\ln\left(-x\right),\ falls\ x<0$$
$$\frac{d}{dx}=\ln\left(\left|x\right|\right)=\ln\left(-x\right)\Rightarrow\frac{1}{-x}\cdot\left(-1\right)=\frac{1}{x}$$

$$z.B.\ \int_{-5}^{-2}\frac{1}{x}\,dx=\ln\left(\left|-x\right|\right)\Big|_{-5}^{-2}=\ln\left(-x\right)\Big|_{-5}^{-2}=\ln\left(2\right)-\ln\left(5\right)$$

Beispiel:

$$\int\frac{x+1}{x^3-5x^2+8x-4}\,dx$$

Schritt 2:

$$\left(x^3-5x^2+8x-4\right):\left(x-1\right)=x^2-4x+4=\left(x-2\right)^2$$
$$\begin{array}{l}
\underline{-x^3-x^2} \\
\qquad -4x^2+8x-4 \\
\underline{-\quad -4x^2+4x} \\
\qquad\qquad\quad 4x-4 \\
\underline{-\qquad\qquad 4x-4} \\
\qquad\qquad\qquad\quad 0
\end{array}$$

$$x^3-5x^2+8x-4=\left(x-1\right)\cdot\left(x-2\right)^2\rightarrow Vielfachheit\ 2\ beachten!!$$

Schritt 3:

$$\frac{x+1}{x^3-5x^2+8x-4}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}\Bigg|\cdot Nenner$$

Anmerkung: Falls im Nenner-Polynom eine Nullstelle x_0 mit der Vielfachheit n auftritt, treten in der Partialbruchzerlegung die Summanden

$$\frac{A_1}{x-x_0},\frac{A_2}{(x-x_0)^2},\ldots,\frac{A_n}{(x-x_0)^n}\text{ auf.}$$

Schritt 4:

$$x+1=A\cdot(x-2)^2+B\cdot(x-1)\cdot(x-2)+C\cdot(x-1)$$

Nullstellen einsetzen:

$$\begin{aligned}
x=1:&\quad 2=A\\
x=2:&\quad 3=C\\
x=0:&\quad 1=4A+2B-C\\
&\quad 1=8+2B-3\Rightarrow B=-2
\end{aligned}$$

$$\frac{x+1}{x^3-5x^2+8x-4}=\frac{2}{x-1}+\frac{-2}{x-2}+\frac{3}{(x-2)^2}$$

Schritt 5:

$$\begin{aligned}
\int\frac{2}{x-1}-\frac{2}{x-2}+\frac{3}{(x-2)^2}&=2\cdot\int\frac{1}{x-1}dx-2\cdot\int\frac{2}{x-2}dx+3\cdot\int\frac{1}{(x-2)^2}dx\\
&=2\cdot\ln(|x-1|)-2\cdot\ln(|x-2|)+3\frac{(x-2)^{-1}}{-1}+c\\
&=\ln(x-1)^2-\ln(x-2)^2-\frac{3}{x-2}+c
\end{aligned}$$

Beispiel:

$$\int_2^3\frac{x^4-2x^3+4x^2+4}{x^6+4x^4+4x^2}dx=?$$

Schritt 2:

$$x^6+4x^4+4x^2=x^2\left(x^4+4x^2+4\right) \qquad SUB:\ z=x^2$$

$$\Rightarrow z^2+4z+4=0 \Rightarrow z_{1,2}=-2\pm\sqrt{4-4}=-2$$

$$\Rightarrow z^2+4z+4=\left(z+2\right)^2 \Rightarrow \left(x^2+2\right)^2$$

$$\Rightarrow x^6+4x^4+4x^2=x^2\cdot\left(x^2+2\right)^2$$

Schritt 3:

$$\frac{x^4-2x^3+4x^2+4}{x^6+4x^4+4x^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+2}+\frac{Ex+F}{\left(x^2+2\right)^2}$$

Schritt 4:

$$\begin{aligned}x^4-2x^3+4x^2+4&=Ax\cdot\left(x^2+2\right)^2+B\cdot\left(x^2+2\right)^2+\left(Cx+D\right)\cdot x^2\cdot\left(x^2+2\right)+\left(Ex+F\right)\cdot x^2\\&=Ax\cdot\left(x^4+4x^2+4\right)+B\cdot\left(x^4+4x^2+4\right)+\left(Cx+D\right)\cdot\left(x^4+4x^2+4\right)+Ex^3+Fx^2\\&=Ax^5+Ax^3+4Ax+Bx^4+4Bx^2+4B+Cx^5+Dx^4+2Cx^3+2Dx^2+Ex^3+Fx^2\\&=\left(A+C\right)\cdot x^5+\left(B+D\right)\cdot x^4+\left(4A+2C+E\right)\cdot x^3+\left(4B+2D+F\right)\cdot x^2+4Ax+4B\end{aligned}$$

Koeffizienten-Vergleich:

$$\begin{array}{llllll l}
A+ & & C & & & & =0\Rightarrow C=0\\
& B+ & & D & & & =1\Rightarrow D=0\\
4A & & +2C & & +\ E & & =-2\Rightarrow E=-2\\
& 4B & & +2D & & +\ F & =4\Rightarrow F=0\\
4A & & & & & & =0\Rightarrow A=0\\
& 4B & & & & & =4\Rightarrow B=1
\end{array}$$

$$\frac{x^4-2x^3+4x^2+4}{x^6+4x^4+4x^2}=\frac{1}{x^2}-\frac{2x}{\left(x^2+2\right)^2}$$

Schritt 5:

$$\int_2^3\frac{1}{x^2}-\frac{2x}{\left(x^2-2\right)^2}\,dx=\int_2^3\frac{1}{x^2}\,dx-\int_2^3\frac{2x}{\left(x^2+2\right)^2}\,dx=-\frac{1}{x}\Big|_2^3-\int_6^{11}\frac{1}{y^2}\,dy \quad \left(SUB:\ y=x^2+2\right)$$

$$=-\frac{1}{3}+\frac{1}{2}+\frac{1}{y}\Big|_6^{11}=\frac{1}{6}+\frac{1}{11}-\frac{1}{6}=\frac{1}{11}$$

Hilfsfunktionen:

$$\left(\tan(x)\right)'=\left(\frac{\sin(x)}{\cos(x)}\right)'=\frac{\cos^2(x)-\sin(x)\cdot\sin(x)}{\cos^2(x)}=\frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}=1+\frac{\sin^2(x)}{\cos^2(x)}=1+\tan^2(x)$$

$$\tan\left(\arctan(x)\right)=x$$

$$\left(1+\tan^2\left(\arctan(x)\right)\right)\cdot\arctan'(x)=1$$

$$\left(1+x^2\right)\cdot\arctan'(x)=1$$

$$\frac{d}{dx}\arctan(x)=\frac{1}{1+x^2}$$

Beispiel:

$$\int\frac{1}{x^2+x+1}dx=?$$

Suche der Nullstellen:

$$x^2+x+1=0\Rightarrow x_{1,2}=-\frac{1}{2}\pm\sqrt{\frac{1}{4}-1}\notin\mathbb{R}$$

Integration:

$$\int\frac{1}{x^2+x+1}dx=\int\frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}dx=\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx=\int\frac{1}{\frac{3}{4}\cdot\left(\frac{4}{3}\cdot\left(x+\frac{1}{2}\right)^2+1\right)}dx$$

$$=\frac{4}{3}\cdot\int\frac{1}{\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)^2+1}dx$$

$$SUB:\quad y=\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\qquad\frac{dy}{dx}=\frac{1}{1+y^2}\cdot\frac{\sqrt{3}}{2}\qquad dx=\frac{\sqrt{3}}{2}dy$$

$$=\int\frac{1}{x^2+x+1}dx=\frac{4}{3}\cdot\int\frac{1}{1+y^2}\cdot\frac{\sqrt{3}}{2}dy=\frac{2}{\sqrt{3}}\cdot\int\frac{1}{1+y^2}dy$$

$$=\frac{2}{\sqrt{3}}\cdot\arctan(y)+c=\frac{2}{\sqrt{3}}\cdot\arctan\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)+c$$